MAT290F - ADVANCED ENGINEERING MATHEMATICS
QUIZ 2 Solution Outline, Section TUT07-08

1. Find and sketch the region of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n+1)^n}{(n+2)^n}\,(z-5+i)^{2n}.$$

The n^{th} term in the series is

$$c_n = \frac{(n+1)^n}{(n+2)^n}\,(z-5+i)^{2n}.$$

Applying the ratio test, we get

$$\begin{aligned}\left|\tfrac{c_{n+1}}{c_n}\right| &= \left|\tfrac{(n+2)^{n+1}}{(n+3)^{n+1}}\tfrac{(n+2)^n}{(n+1)^n}\,(z-5+i)^2\right| \\ &= \left|(1-\tfrac{1}{n+3})^{n+1}\right|\left|(1+\tfrac{1}{n+1})^n\right|\left|(z-5+i)^2\right|.\end{aligned}$$

Since $\lim_{n\to\infty}(1+\frac{k}{n})^n = e^k$, we have

$$\left|\frac{c_{n+1}}{c_n}\right| \to |z-5+i|^2$$

So the region of convergence is the open disk center $5-i$, radius 1.

2. Find all solutions of $\sin(z) = 2$.

Since

$$\sin(z) = \frac{\mathrm{e}^{\mathrm{iz}} - \mathrm{e}^{-\mathrm{iz}}}{2i}$$

we have

$$\begin{aligned} \mathrm{e}^{\mathrm{iz}} - \mathrm{e}^{-\mathrm{iz}} &= 4i\,. \\ (\mathrm{e}^{\mathrm{iz}})^2 - 4\mathrm{i}\mathrm{e}^{\mathrm{iz}} - 1 &= 0\,. \end{aligned}$$

or

$$\mathrm{e}^{\mathrm{iz}} = \frac{4\mathrm{i} \pm \sqrt{-16+4}}{2} = \mathrm{i}(2 \pm \sqrt{3})\,.$$

Case I:

$$\mathrm{e}^{\mathrm{iz}} = \mathrm{e}^{\mathrm{ix-y}} = (2+\sqrt{3})\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+2\pi\mathrm{n})}, \qquad \mathrm{n} = 0, \pm 1, \pm 2, \ldots\,.$$

So

$$\begin{aligned} x &= \frac{\pi}{2} + 2\pi n \\ y &= -\ln(2+\sqrt{3})\,. \end{aligned}$$

Case II:

$$\mathrm{e}^{\mathrm{iz}} = \mathrm{e}^{\mathrm{ix-y}} = (2-\sqrt{3})\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+2\pi\mathrm{n})}, \qquad \mathrm{n} = 0, \pm 1, \pm 2, \ldots\,.$$

So

$$\begin{aligned} x &= \frac{\pi}{2} + 2\pi n \\ y &= -\ln(2-\sqrt{3})\,. \end{aligned}$$

3. Let $f(z) = \frac{2z-3}{z}$,

(a) Compute $\int_\Gamma f(z)dz$ if Γ denotes the clockwise semicircle of radius 2 about the origin from -2 to 2.

(b) Compute $\int_\Gamma f(z)dz$ if Γ denotes the counter-clockwise semicircle of radius 2 about the origin from -2 to 2.

(a)

$$f(z) = \frac{2z-3}{z} = 2 - \frac{3}{z}.$$

So

$$\int_\Gamma f(z)dz = \int_\Gamma (2 - \frac{3}{z})dz = \int_\Gamma 2dz - \int_\Gamma \frac{3}{z}dz.$$

The first term $\int_\Gamma 2dz = 2z|_{-2}^{2} = 8$.

Parameterize Γ by $z = 2e^{i(\pi - t)}, \qquad 0 \le t \le \pi$.

Then $z' = -2ie^{i(\pi-t)}$. So

$$\int_\Gamma \frac{3}{z}dz = \int_0^\pi \frac{3}{2e^{i(\pi-t)}}(-2ie^{i(\pi-t)})dt = \int_0^\pi (-3i)dt = -3\pi i,$$

and then

$$\int_\Gamma f(z)dz = 8 + 3\pi i.$$

(b) The first term $\int_\Gamma 2dz = 2z|_{-2}^{2} = 8$.

Parameterize Γ by $z = 2e^{it}, \qquad \pi \le t \le 2\pi$.

Then $z' = 2ie^{it}$. So

$$\int_\Gamma \frac{3}{z}dz = \int_\pi^{2\pi} \frac{3}{2e^{it}}(2ie^{it})dt = \int_\pi^{2\pi} (3i)dt = 3\pi i,$$

and then

$$\int_\Gamma f(z)dz = 8 - 3\pi i.$$

4. Let $f(z) = \frac{1}{z^2}$. Then show that

$$\left| \int_\Gamma f(z)dz \right| \leq 3\sqrt{2}$$

where Γ denotes the curve made up of two straight line segments from $z = i$ to $z = 1$, and from $z = 1$ to $z = 2 + i$.

Let Γ_1 denote the straight line segment from $z = i$ to $z = 1$, and let Γ_2 denote the straight line segment from $z = 1$ to $z = 2 + i$. Then

$$\left| \int_\Gamma f(z)dz \right| = \left| \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz \right| \leq \left| \int_{\Gamma_1} f(z)dz \right| + \left| \int_{\Gamma_2} f(z)dz \right|.$$

$\left|\frac{1}{z^2}\right| \leq 2$ for all z on Γ_1 and $\left|\frac{1}{z^2}\right| \leq 1$ for all z on Γ_2. The length of Γ_1 is $|1 - i| = \sqrt{2}$, and the length of Γ_2 is $|2 + i - 1| = \sqrt{2}$.

So

$$\left| \int_\Gamma f(z)dz \right| \leq 2 \times \sqrt{2} + 1 \times \sqrt{2} = 3\sqrt{2}.$$