MAT290F - ADVANCED ENGINEERING MATHEMATICS
QUIZ 1 Solution Outline, Section TUT07-08

1. (a) Write the definition of the Laplace transform. By direct integration, derive the Laplace transform of the function

$$f(t) = cosh(3t)$$

(b) For what range of s is the transform valid?

(a)

$$\begin{aligned} F(s) &= \int_0^\infty f(t)\mathrm{e}^{-st}dt \\ &= \int_0^\infty cosh(3t)\ \mathrm{e}^{-st}dt \qquad \textbf{[1 mark]} \\ &= \int_0^\infty \frac{(\mathrm{e}^{3t} + \mathrm{e}^{-3t})}{2}\ \mathrm{e}^{-st}dt \\ &= \left.\frac{\mathrm{e}^{-(s-3)t}}{-2(s-3)}\right|_0^\infty + \left.\frac{\mathrm{e}^{-(s+3)t}}{-2(s+3)}\right|_0^\infty \\ &= \frac{1}{2(s-3)} + \frac{1}{2(s+3)} \\ &= \frac{s}{s^2-9} \qquad \textbf{[3 marks]} \end{aligned}$$

(b) $s > 3$ **[1 mark]**

2. Find the Laplace transform of a periodic function whose definition in one period is:

$$f(t) = \begin{cases} t, & 0 < t < \text{a} \\ 0, & a < t < 2a \end{cases}$$

$$F(s) = \frac{1}{1 - \mathrm{e}^{-sT}} \int_0^T f(t)\mathrm{e}^{-st} dt \; ; \qquad T = 2a \qquad \textbf{[1 mark]}$$

$$\begin{aligned}
\int_0^T f(t)\mathrm{e}^{-st} dt &= \int_0^{2a} f(t)\mathrm{e}^{-st} dt \\
&= \int_0^a t\ \mathrm{e}^{-st} dt + \int_a^{2a} 0.\ \mathrm{e}^{-st} dt \\
&= \left. \frac{t\mathrm{e}^{-st}}{-s} \right|_0^a + \int_0^a \frac{\mathrm{e}^{-st}}{s} dt \\
&= \frac{-a\mathrm{e}^{-as}}{s} - \left. \frac{\mathrm{e}^{-st}}{s^2} \right|_0^a \\
&= \frac{-a\mathrm{e}^{-as}}{s} + \frac{1 - \mathrm{e}^{-as}}{s^2} \qquad \textbf{[2 marks]} \\
F(s) &= \frac{1}{1 - \mathrm{e}^{-2as}} \times \frac{1 - as\mathrm{e}^{-as} - \mathrm{e}^{-as}}{s^2} \\
&= \frac{1 - as\mathrm{e}^{-as} - \mathrm{e}^{-as}}{s^2(1 - \mathrm{e}^{-2as})} \qquad \textbf{[2 marks]}
\end{aligned}$$

3. Solve the initial value problem

$$y''(t) + 2y'(t) + y(t) = t\mathrm{e}^{-t}, \quad y(0) = 1, \quad y'(0) = -2$$

$$\begin{aligned}
(s^2Y(s) - sy(0) - y'(0)) + 2(sY(s) - y(0)) + Y(s) &= \frac{1}{(s+1)^2} \\
s^2Y(s) - s + 2 + 2sY(s) - 2 + Y(s) &= \frac{1}{(s+1)^2} \quad \textbf{[1 mark]} \\
(s^2 + 2s + 1)Y(s) &= \frac{1}{(s+1)^2} + s \\
Y(s) &= \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2} \\
&= \frac{1}{(s+1)^4} + \frac{s+1-1}{(s+1)^2} \\
&= \frac{1}{(s+1)^4} + \frac{1}{s+1} - \frac{1}{(s+1)^2}
\end{aligned}$$

$$y(t) = (\frac{1}{6}t^3 + 1 - t)\mathrm{e}^{-t} \quad \textbf{[4 marks]}$$

4. Find the inverse Laplace transform of the following function by convolution theorem.

$$F(s) = \frac{s}{s^3 + 2s^2 + 4s + 8}$$

$$\begin{aligned}
F(s) &= \frac{s}{(s+2)(s^2+4)} \\
f(t) &= \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] * \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] \\
&= \mathrm{e}^{-2t} * \cos(2t) \\
&= \int_0^t \mathrm{e}^{-2(t-\tau)} \cos(2\tau) d\tau \\
&= \mathrm{e}^{-2t} \int_0^t \mathrm{e}^{2\tau} \cos(2\tau) d\tau \qquad \textbf{[2 marks]}
\end{aligned}$$

Let

$$\begin{aligned}
y(t) &= \int_0^t \mathrm{e}^{2\tau} \cos(2\tau) d\tau \\
&= \frac{1}{2}\mathrm{e}^{2t} \sin(2t) - \int_0^t \mathrm{e}^{2\tau} \sin(2\tau) d\tau \\
&= \frac{1}{2}\mathrm{e}^{2t} \sin(2t) - \left[\frac{1}{2} - \frac{1}{2}\mathrm{e}^{2t} \cos(2t) + y(t)\right]
\end{aligned}$$

So

$$\begin{aligned}
y(t) &= \frac{1}{4}\mathrm{e}^{2t} \sin(2t) + \frac{1}{4}\mathrm{e}^{2t} \cos(2t) - \frac{1}{4} \\
f(t) &= \frac{1}{4}\left[\sin(2t) + \cos(2t) - \mathrm{e}^{-2t}\right]. \qquad \textbf{[3 marks]}
\end{aligned}$$

5. Find the inverse Laplace transform of the function

$$F(s) = \frac{1}{(s+a)(1-\mathrm{e}^{-ks})}$$

where a and k are positive constants. Hint: Use the geometric series formula

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}\,, \qquad -1 < r < 1\,.$$

For $k > 0$ and $s > 0$, $\mathrm{e}^{-ks} < 1$ so we have

$$\frac{1}{1-\mathrm{e}^{-ks}} = 1 + \mathrm{e}^{-ks} + \mathrm{e}^{-2ks} + \mathrm{e}^{-3ks} + \ldots \qquad \textbf{[1 mark]}$$

$$F(s) = \frac{1}{s+a}(1 + \mathrm{e}^{-ks} + \mathrm{e}^{-2ks} + \mathrm{e}^{-3ks} + \ldots) \qquad \textbf{[1 mark]}$$

$$\begin{aligned}\mathcal{L}^{-1}[F(s)] &= \mathrm{e}^{-at}H(t) + \mathrm{e}^{-a(t-k)}H(t-k) + \mathrm{e}^{-a(t-2k)}H(t-2k) + \\ &\quad \mathrm{e}^{-a(t-3k)}H(t-3k) + \ldots \qquad \textbf{[3 marks]}\end{aligned}$$