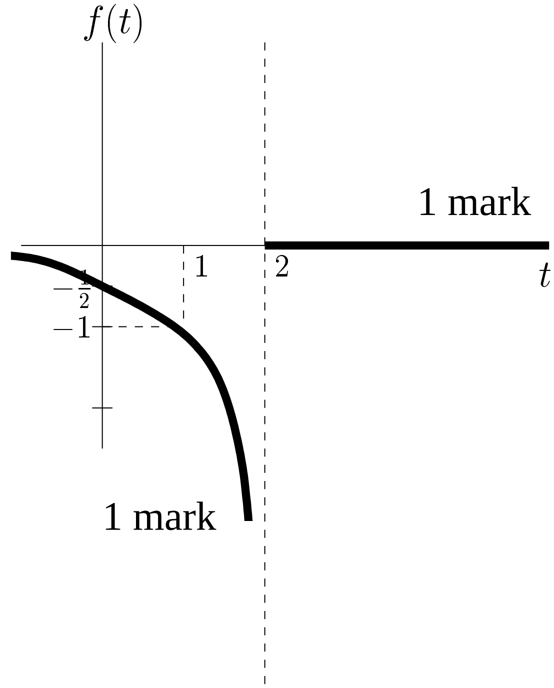
MAT290F - ADVANCED ENGINEERING MATHEMATICS
QUIZ 1 Solution Outline, Section TUT05-06

1. Plot the function

$$f(t) = \begin{cases} \frac{1}{t-2}, & t < 2 \\ 0, & t \geq 2\,. \end{cases}$$

   Does the Laplace transform exist? Does the Existence theorem for Laplace transforms tell you anything?



The Laplace transform does not exist because the integral is not defined. [**1 mark**]

No. The Existence theorem tells us that the Laplace transform exists if the function is piecewise continuous and of exponential order. Because this function neither piecewise continuous nor of exponential order, the theorem does not tell us anything. [**2 marks**]

2. Find the inverse Laplace transform $f(t)$ of the function

$$F(s) = \frac{2}{(s^2+8)s^3}$$

$$\begin{aligned} F(s) &= \frac{2}{(s^2+8)s^3} \\ &= \frac{as+b}{s^2+8} + \frac{cs^2+ds+e}{s^3} \\ &= \frac{\frac{1}{32}s}{s^2+8} + \frac{\frac{-1}{32}s^2+\frac{1}{4}}{s^3} \\ &= \frac{1}{32}\frac{s}{(s^2+8)} - \frac{1}{32s} + \frac{1}{4s^3} \end{aligned}$$

[2 marks]

$$f(t) = -\frac{1}{32} + \frac{t^2}{8} + \frac{1}{32}\cos(2\sqrt{2}t), \qquad t \geq 0$$

[3 marks]

3. Using the definition of Laplace transform and Heaviside function, find the Laplace transform of

$$H(t-a)f'(t-a)$$

as a function of $f(t)$ and its Laplace transform.

$$\begin{aligned}
\mathcal{L}[H(t-a)f'(t-a)](s) &= \int_0^\infty H(t-a)f'(t-a)e^{-st}dt && \textbf{[1 mark]} \\
&= \int_a^\infty e^{-st}f'(t-a)dt && \textbf{[2 marks]} \\
&= e^{-sa}\int_0^\infty e^{-s\tau}f'(\tau)d\tau && \text{integ. by parts} \\
&= e^{-sa}[[e^{-s\tau}f(\tau)]_0^\infty + s\int_0^\infty e^{-s\tau}f(\tau)d\tau] && \textbf{[1 mark]} \\
&= e^{-sa}[sF(s) - f(0)] && \textbf{[1 mark]}
\end{aligned}$$

4. Solve the initial value problem

$$y'(t) + 3y(t) + 2\int_0^t y(t)dt = f(t), \quad y(0) = 0,$$

where

$$f(t) = \begin{cases} 0, & t < 2, \\ t^2 - 3t + 2, & t \geq 2. \end{cases}$$

$$\begin{aligned} f(t) &= \left[t^2 - 3t + 2\right] H(t-2) \\ &= \left[(t-2)^2 + (t-2)\right] H(t-2)\,. \end{aligned}$$

$$F(s) = e^{-2s}\left(\frac{2}{s^3} + \frac{1}{s^2}\right).$$

[1 mark]

Take the transform of each side of the given equation to get

$$sY(s) - y(0) + 3Y(s) + \frac{2}{s}Y(s) = F(s).$$

$$\begin{aligned} Y(s) &= \tfrac{s}{(s+1)(s+2)}F(s) \\ &= \tfrac{s}{(s+1)(s+2)}\left(e^{-2s}\tfrac{2}{s^3} + e^{-2s}\tfrac{1}{s^2}\right) \\ &= e^{-2s}\left(\tfrac{1}{s^2} - \tfrac{1}{s} + \tfrac{1}{(s+1)}\right). \end{aligned}$$

[3 marks]

$$y(t) = \left(t - 3 + e^{-(t-2)}\right) H(t-2).$$

[1 mark]

5. Using the convolution theorem, find the inverse Laplace transform $f(t)$ of the function

$$F(s) = \frac{e^{-4s}}{s(s^2 + a^2)} .$$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[\frac{e^{-4s}}{s}\right] * \mathcal{L}^{-1}\left[\frac{1}{a}\frac{a}{s^2+a^2}\right] \\ &= \frac{1}{a}H(t-4) * \sin(at) \qquad \textbf{[2 marks]} \\ &= \frac{1}{a}\int_4^t \sin a(t-\tau)d\tau \\ &= \frac{1}{a^2}\left[1 - \cos a(t-4)\right] H(t-4) \qquad \textbf{[3 marks]} \end{aligned}$$