MAT290F - ADVANCED ENGINEERING MATHEMATICS
QUIZ 1 Solution Outline, Section TUT01-04

1. Write the definition of the Laplace transform. By any method find the Laplace transform $F(s)$ of the function

$$f(t) = \int_0^t e^{-a\tau} \sin(b\tau) d\tau$$

$$\begin{aligned} F(s) &= \int_0^\infty f(t) \mathrm{e}^{-st} dt \qquad \textbf{[1 mark]} \\ F(s) &= \frac{1}{s} \mathcal{L} \left[\mathrm{e}^{-at} \sin(bt) \right] \\ &= \frac{1}{s} \frac{b}{\left[(s+a)^2 + b^2 \right]} \qquad \textbf{[4 marks]} \end{aligned}$$

2. Solve the initial value problem

$$y'' + 5y' + 6y = f(t) \qquad\qquad y(0) = y'(0) = 0$$

where

$$f(t) = \begin{cases} -2, & 0 \le t < 3, \\ 0, & t \ge 3. \end{cases}$$

$$f(t) = -2[H(t) - H(t-3)]$$

$$F(s) = -2\left[\frac{1}{s} - \frac{e^{-3s}}{s}\right] \qquad \textbf{[1 mark]}$$

Take the transform of each side of the given equation to get

$$s^2Y(s) + 5sY(s) + 6Y(s) = F(s) \qquad \textbf{[1 mark]}$$

$$\begin{aligned} Y(s) &= \frac{-2}{s(s+2)(s+3)} + \frac{2e^{-3s}}{s(s+2)(s+3)} \\ &= \left[\frac{1}{(s+2)} - \frac{2}{3}\frac{1}{(s+3)} - \frac{1}{3s}\right](1 - e^{-3s}) \qquad \textbf{[2 marks]} \end{aligned}$$

$$y(t) = \left[e^{-2t} - \frac{2}{3}e^{-3t} - \frac{1}{3}\right]H(t) - \left[e^{-2(t-3)} - \frac{2}{3}e^{-3(t-3)} - \frac{1}{3}\right]H(t-3) \qquad \textbf{[1 marks]}$$

3. Solve the integral equation

$$y(t) = \sin(5t) - 6 \int_0^t y(t-\lambda)\cos(5\lambda)d\lambda.$$

Transform each side of the equation using the Convolution Theorem, to obtain

$$Y(s) = \frac{5}{s^2+25} - 6Y(s)\frac{s}{s^2+25}.$$

[2 marks]

Solving for $Y(s)$, we have

$$Y(s) = \left[1 + \frac{6s}{s^2+25}\right]^{-1} \frac{5}{s^2+25} = \frac{5}{4}\frac{4}{(s+3)^2+4^2}$$

so

$$y(t) = \frac{5}{4}e^{-3t}\sin(4t).$$

[3 marks]

4. Given that $F(s) = \mathcal{L}[f(t)]$, show that

$$\mathcal{L}\left[e^{\frac{bt}{a}} f(\frac{t}{a})\right] = aF(as - b).$$

$$\mathcal{L}\left[e^{\frac{bt}{a}} f(\frac{t}{a})\right] = \int_0^\infty e^{-st} e^{\frac{bt}{a}} f(\frac{t}{a}) dt \qquad \textbf{[2 marks]}$$

Let $\sigma = \frac{t}{a}$, then

$$\mathcal{L}\left[e^{\frac{bt}{a}} f(\frac{t}{a})\right] = a \int_0^\infty e^{-sa\sigma} e^{b\sigma} f(\sigma) d\sigma$$

And let $s' = sa$, then

$$\mathcal{L}\left[e^{\frac{bt}{a}} f(\tfrac{t}{a})\right] = a \int_0^\infty e^{-s'\sigma} e^{b\sigma} f(\sigma) d\sigma = a\mathcal{L}\left[e^{bt} f(t)\right](s')$$

[3 marks]

$$= aF(s' - b) = aF(as - b).$$

5. Show that the function

$$f(t) = 2te^{t^2}\cos\left(e^{t^2}\right)$$

has a Laplace transform even though $f(t)$ is not of exponential order.

$f(t)$ is continuous on $[0,\infty)$ but not of exponential order. However, the Laplace transform of $f(t)$ is

$$\begin{aligned}\mathcal{L}[f](s) &= \int_0^\infty e^{-st}2te^{t^2}\cos\left(e^{t^2}\right)dt \\ &= \left[e^{-st}\sin\left(e^{t^2}\right)\right]_0^\infty + s\int_0^\infty e^{-st}\sin\left(e^{t^2}\right)dt \\ &= -\sin(1) + s\mathcal{L}\left[\sin\left(e^{t^2}\right)\right]\end{aligned}$$

[3 marks]

The function $\sin\left(e^{t^2}\right)$ is continuous on $[0,\infty)$, and there are numbers $M = 1, b = 0$ such that

$$\left|\sin\left(e^{t^2}\right)\right| \leq 1 = Me^{bt}.$$

Thus, $\mathcal{L}[\sin\left(e^{t^2}\right)]$ exists, and therefore the function $f(t) = 2te^{t^2}\cos\left(e^{t^2}\right)$ has a Laplace transform.

[2 marks]