Name: ______________________________ Student ID: ______________

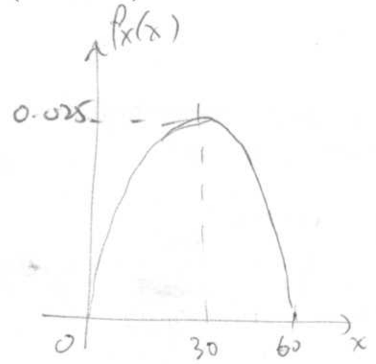
QUIZ 2, ECE302, Fall 2003
Time: 50 minutes. Total Marks: 40.
One single-sided 8.5x11" formula sheet allowed, calculators allowed.

1. X is the time (in minutes) that a student would take to write this quiz if there were no time limit. Suppose X has the following pdf: $f_X(x) = c(60x - x^2)$ if $0 \le x \le 60$ and $f_X(x) = 0$ otherwise.

a) Find c, sketch the pdf, and find E[X]. (5 marks)

$$\int_0^{60} c(60x - x^2)\,dx = 1$$
$$\Rightarrow c\left(30x^2 - \frac{x^3}{3}\right)\Big|_0^{60} = 1$$
$$\Rightarrow c\left(30(60)^2 - \frac{(60)^3}{3}\right) = 1$$
$$\Rightarrow c(1000)\left(108 - \frac{216}{3}\right) = 1$$
$$\Rightarrow c = \frac{1}{36000}.$$

2 marks

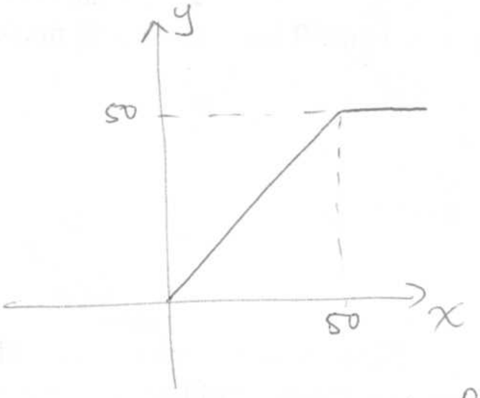


3 marks

$E(X) = 30$ 1 mark because of symmetry of pdf about $x = 30$.

72
3)216
108
−72
36

b) Students must hand in their quizzes after 50 minutes, so the actual writing time cannot be greater than 50 minutes. Let Y be the actual writing time. Find Y as a function of X and sketch it. Find the pdf of Y and sketch it. (10 marks)

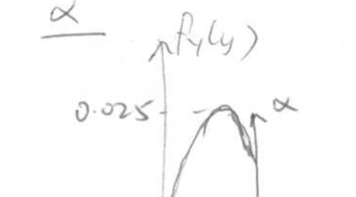


$$Y = \begin{cases} X & 0 \le X \le 50 \\ 50 & 50 < X \le 60 \end{cases}$$

$F_Y(y) = P(Y \le y) = P(X \le y)\ (0 \le y \le 50)$
$= F_X(y)$

$P(Y = 50) = P(50 < X \le 60) = \int_{50}^{60} c(60x - x^2)\,dx$
$= \alpha$

$$\Rightarrow f_Y(y) = \cancel{\begin{cases} f_X(y) & 0 \le y \le 50 \\ \alpha\delta(y-50) \end{cases}}$$
$$= f_X(y)[u(y) - u(y-50)] + \alpha\delta(y-50)$$



c) Find E[Y] (5 marks).

$$E(Y) = \int_0^{50} c\cdot y(60y - y^2)\,dy + 50P(Y=50)$$
$$= c\left[20y^3 - \frac{1}{4}y^4\right]_0^{50} + 50\int_{50}^{60} \frac{1}{36000}\cdot(60x - x^2)\,dx$$
$$= c[937500) + \frac{5}{3600}\cdot\left(30x^2 - \frac{1}{3}x^3\right)\Big|_{50}^{60}$$
$$= 29.745.$$

2. Suppose we don't know the distribution of the time Y a student needs to write the quiz, but we know that $E[Y] = 40$ and $VAR[Y] = 400$. We can use the Markov inequality ($P[X \geq a] \leq E[X]/a, X \geq 0, a > 0$) and the Chebyshev inequality ($P[|X - E[X]| \geq a] \leq VAR[X]/a^2$) to find bounds on certain probabilities.

a) Using the Markov inequality, find an upper bound on the probability that $Y \geq 50$. (4 marks)

$P(Y \geq 50) \leq \frac{40}{50} = 0.8$

b) Using the Chebyshev inequality, find a *lower* bound on the probability that Y is *within* 30 minutes of the mean time (40 minutes). (8 marks)

Chebyshev inequality yields

~~P[|X~~ $\underbrace{P[|Y-40| \geq 30]}_{A} \leq \frac{400}{\cancel{1600}\ 900} = \cancel{0.25}\ 4/9.$ } 3 marks

$A^c = \{|Y-40| \leq 30\}$ (the desired event) } 3 marks

$P(A) + P(A^c) = 1 \Rightarrow P(A^c) = 1 - P(A), \quad P(A) \leq \cancel{0.25}\ 4/9$

$\Rightarrow P(A^c) \geq \cancel{0.75}\ 1 - \frac{4}{9} = \frac{5}{9}.$ } Ans 2 marks

c) For $a < E[X]$, the Markov inequality guarantees that $P[X \geq a] \leq c$, where $c > 1$. Find a better upper bound, *i.e.*, find a value d, where $P[X \geq a] \leq d$ and $d < c$. Hint: This is trivial. (3 marks)

$d = 1$

d) The Markov inequality guarantees that $P[X \geq E[X]] \leq 1$. Show that for symmetric pdfs, $P[X \geq \cancel{q}\ E(X)] = 1/2$. Hint: $f_X(x)$ is symmetric if $f_X(q-x) = f_X(q+x)$ for some number q. (5 marks)

$P(X \geq E(X)) = \int_{\mu}^{\infty} f_X(x)\,dx$

where $\mu = E(X)$.

But $f_X(x)$ is symmetric about $x = \mu$, and therefore

$\int_{-\infty}^{\mu} f_X(x)\,dx = \int_{\mu}^{\infty} f_X(x)\,dx$

Combining this with the fact that $\int_{-\infty}^{\mu} f_X(x)\,dx + \int_{\mu}^{\infty} f_X(x)\,dx = 1$

gives $\int_{\mu}^{\infty} f_X(x)\,dx = 1/2$.