NAME: ______________________ STUDENT NO.: __________

UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Midterm Examination, Oct 24, 2002

ECE 302F — Probability and Applications

Examination Type: Closed book, Calculators allowed, 2-sided 8x11 "cheat sheet" allowed
Examiners: P. Aarabi, B. J. Frey, T. J. Lim

Instructions

- The last page gives some useful formulas.
- This examination paper consists of ten [10] pages (including this one). Please make sure that you have a complete paper.
- Write your name and student number on each page, and enter the first letter of your last name in the box at the bottom of this page.
- Answer **all** six [6] questions. The value of each question is noted on each page. A total of **120 marks** is available.
- Answer each question directly on the examination paper, using the back of each page if necessary. Please indicate clearly where your work can be found.
- Show your work! Show all steps and present all results clearly. State any assumptions that you make.
- If we can't read it, we can't mark it. Please be neat, and **use ink**.
- Time: 1.5 hours.

Enter the first letter of

[ ]

your last name here.

EXAMINER'S REPORT

1.		/20
2.		/20
3.		/20
4.		/20
5.		/20
6.		/20
Total:		/120

Total 20 marks

1. This question tests your ability to solve probability modeling problems in MATLAB.

4 marks

(**a**) The command `X = rand` randomly assigns a value to `X`. Specify the sample space of `X` and give an expression for its pdf or pmf.

4 marks

(**b**) For the command `X = rand > 0.2`, specify the sample space of `X` and its pdf or pmf.

4 marks

(**c**) For the command `X = 7 * randn + 2`, specify the sample space of `X` and its pdf or pmf.

3 marks

(**d**) Write down a MATLAB statement for drawing 100 Bernoulli random variables with $P(Y = 1) = .6$ and $P(Y = 0) = .4$ and placing them in the vector `Y`.

5 marks

(e) Suppose you perform 3 computer experiments, in which you draw $n = 10$, $n = 1000$ and $n = 100,000$ Gaussian random variables with the same mean and standard deviation. In each case, you construct a histogram. For each value of n, sketch a plausible histogram – the histograms need not be precise, but they should convey the main observations you made in the computer laboratory.

Total 20 marks

2. The continuous random variable X has a pdf $f_X(x) = \frac{3}{2}x^2$ for $-1 < x < 1$. Now, we define the random variable Y such that $Y = X^3 + b$.

10 marks

(**a**) What is the pdf of Y, $f_Y(y)$? (Note: Once you have $f_Y(y)$ in terms of $f_X(x)$, make sure to substitute the actual definition of $f_X(x)$ above).

5 marks

(**b**) What is the mean of Y?

5 marks

(**c**) What is the variance of Y?

NAME: ______________________________ STUDENT NO.: __________

Total 20 marks

3. Let X be a random variable with pmf

$$P(X = k) = \begin{cases} \alpha & k = 0, 2, 4, \ldots, 2(n-1) \\ 0 & \text{otherwise} \end{cases},$$

where n is a positive integer.

5 marks

(a) Find an expression for α in terms of n.

5 marks

(b) What is $\mathrm{E}[X]$?

5 marks

(c) For $n = 4$, what is $\mathrm{E}[X^2]$?

5 marks

(d) What is the variance of X?

Total 20 marks

4. The Canadian Space Agency (CSA) has hired consulting firm D. Fective and Associates to design and build a detection system that counts the number of subatomic particles striking the sensor of a deep space probe every 24 hours. At the beginning of each 24-hour interval, the counter is reset and after the interval is finished, the probe transmits back to Earth the number of particles counted. Unfortunately, the counting system is faulty, and once the counter reaches 5, it does not increase. For example, if the true number of particles is 10, the counter still shows 5.

8 marks

(a) Assuming the true number of counts Y follows a Poisson distribution with mean $\alpha = 3$, specify the sample space S_X and the pmf (numerically) of the number of counts sent back to Earth, X.

4 marks

(b) Sketch the cdf of X.

8 marks (**c**) Back on Earth, a new firm, ECE 302 Solutions, has been hired to compute the distribution over the true counts Y, given that the number of counts received on Earth is $X = 5$. Write down the event in S_Y that is equivalent to $X = 5$, and derive an expression for $P(Y = k|X = 5)$, $k \in S_Y$.

Total 20 marks

5. The Canadian Space Agency is so impressed by the work of ECE 302 Solutions, that they hire the firm to analyze one of their defective deep-space communication systems. For space station A to send information to space station C, the signal must be relayed by an intermediate station B. Station A sends a 0 with probability ρ_1, or a 1 with probability $1-\rho_1$ to station B. If station B receives a 0, it will send a 0 with probability 1 to station C. However, if station B receives a 1, its over-heated circuitry will occasionally send a 0 with probability ρ_2, but otherwise send a 1 with probability $1-\rho_2$ to station C. *(You are strongly encouraged to use a tree diagram for this question.)*

10 marks

(a) What is the probability that station C receives a 0?

4 marks

(b) What is the probability that station A sends a 0 *and* station C receives a 0?

6 marks

(c) If $0 < \rho_1 < 1$, then for what values of ρ_2 are the events "Station C receives a 0" and "Station A sends a 0" independent?

Total 20 marks

6. After a hard day of work at ECE 302 Solutions, you often like to go home, relax, and perform some experiments with your urn and balls. You have an urn which contains 2 red balls, 1 green ball, and 1 blue ball. You select *one* ball from the urn with uniform probability, note its colour, and put it back in the urn. You perform this experiment a total of 8 times.

10 marks

(a) What is the probability that you selected 7 or more red balls, in total?

10 marks

(b) What is the probability that you selected 1 or more green balls, in total?

List of Formulas

Gaussian pdf: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$,
where $E[X] = \mu$ and $\text{VAR}[X] = \sigma^2$.

Poisson pmf: $P(X = k) = \frac{\alpha^k}{k!} e^{-\alpha}, k = 0, 1, \ldots$
where $\text{E}[X] = \alpha$ and $\text{VAR}[X] = \alpha$.

Geometric pmf: $P(X = k) = p(1-p)^{k-1}$, where $k = 1, 2, \ldots$

Exponential pdf: $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

Binomial pmf: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \ldots, n.$