ECE302F Mid-Term Examination 2001

Time allowed: One and a half (1.5) hours.

Rules:

- One double-sided letter-size aid sheet allowed;
- Non-programmable electronic calculators CAN be used;
- Answer all FIVE (5) questions;
- A list of formulae can be found at the end of this question booklet.

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1. Consider a game with two bags of numbered balls:

   - Bag 1 has four balls labelled '9', and four labelled '10';
   - Bag 2 has one ball labelled '8', three labelled '9', and four labelled '10'.

   The balls are identical except for their labels. There are two players in the game:

   - Player A draws two balls from bag 1 without replacement and adds up the two numbers on the balls to obtain his score;
   - Player B draws two balls from bag 2 without replacement and adds up his two numbers to obtain his score.

   A player wins if his score is larger than the other player's; draws (ties) are possible.

   (a) List all the distinct unordered pairs of balls that can be drawn from bag 1. Do the same for bag 2. (8 marks)

   (b) Let X and Y be the points scored by players A and B respectively. Find the probability mass functions of the random variables X and Y. (12 marks)

2. In a fictitious country, couples have children until they have their first girl and then they stop having children. Assume that the probability of having a girl is p and the probability of having a boy is $1-p$. p is constant and does not depend on the number of children the couple already have. To solve the problems below, you may find the following result useful:

$$\sum_{k=1}^{\infty} \frac{r^{k-1}}{k} = -\frac{\ln(1-r)}{r}, \qquad r < 1$$

(a) Write down the probability that a family will have only one child; write down the probability that a family has four children. What well-known distribution does the random variable **number of children in family** follow? (6 marks)

(b) Derive an expression for the *average fraction* of girls in a family. (Hint: For a family of size 1, the fraction of girls in the family is 1/1; for a family of size 2, the fraction of girls is 1/2; size 3, the fraction is 1/3; etc. Find the average fraction.) (9 marks)

(c) If the probability of giving birth to a girl is equal to the probability of giving birth to a boy (i.e. $p = 1/2$), calculate the average fraction of girls in a family. (2 marks)

(d) Explain **in 20 words or less** why this result makes sense in light of the fact $p = 1/2$. (Hint: If $p = 1/2$, half of all families in the country have only one child – a girl.) (3 marks)

3. Consider the following probability density function

$$f_X(x) = \begin{cases} e^{-x} & 0 \leq x < \infty \\ 0 & -\infty < x < 0 \end{cases} .$$

(a) Find the cumulative distribution function $F_X(x)$. (6 marks)

(b) From the definition above, it is clear that $f_X(x)$ is defined over two regions: (1) from $0 \leq x < \infty$ and (2) from $-\infty < x < 0$. Let $f_Y(y)$ be the pdf derived from the transformation $Y = e^X$. What are the equivalent regions over which $f_Y(y)$ is defined? (6 marks)

(c) Find $f_Y(y)$ for $Y = e^X$. (8 marks)

4. In various applications, it is often necessary to compute the Kurtosis of a Gaussian random variable. The Kurtosis is defined as

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right]$$

where μ is the mean of the random variable X and σ^2 is its variance.

(a) For a zero-mean unit-variance Gaussian random variable, find its Kurtosis. (Hint: Use the characteristic function of a zero-mean unit-variance Gaussian random variable and its derivatives, which are given in the formula list on page 3.) (10 marks)

(b) Let X be a Gaussian random variable with mean μ and variance σ^2. Now, let $Y = \dfrac{X-\mu}{\sigma}$. What is the mean and variance of Y? (5 marks)

(c) Using the results of parts (a) and (b), what is the Kurtosis of a Gaussian random variable with mean μ and variance σ^2? (5 marks)

5. The joint pdf of a 2-element random vector (X_1, X_2) is

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} ce^{-x_1-2x_2} & 0 < x_1 < x_2 < \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) Find c. (5 marks)

(b) Find the marginal pdf of X_1 and the marginal pdf of X_2. (10 marks)

(c) Find $P(X_1 < 2)$. (5 marks)

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List of Formulae

Gaussian pdf: $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\dfrac{(x-\mu)^2}{2\sigma^2}\right]$,
where $E[X] = \mu$ and $\text{var}(X) = \sigma^2$.

Geometric pmf: $P(X = k) = p(1-p)^{k-1}$, where $k = 1, 2, \ldots$

Exponential pdf: $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

Binomial pmf: $P(X = k) = \dbinom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \ldots, n.$

The characteristic function of a zero-mean unit-variance Gaussian random variable is $\Phi(\omega) = e^{-\omega^2/2}$, and its first four derivatives are

$$\begin{aligned}
\Phi'(\omega) &= -\omega e^{-\omega^2/2} \\
\Phi''(\omega) &= \omega^2 e^{-\omega^2/2} - e^{-\omega^2/2} \\
\Phi'''(\omega) &= 3\omega e^{-\omega^2/2} - \omega^3 e^{-\omega^2/2} \\
\Phi^{\text{IV}}(\omega) &= 3e^{-\omega^2/2} - 6\omega^2 e^{-\omega^2/2} + \omega^4 e^{-\omega^2/2}
\end{aligned}$$