UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, DECEMBER 2002

Third Year Electrical and Computer Engineering Program

ECE302H1F – PROBABILITY AND APPLICATIONS

Exam Type: C

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Instructions:

- Answer all FIVE (5) questions;
- There are FOUR (4) pages in this question booklet;
- All questions carry equal marks.

List of Formulae

Gaussian pdf: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$,
where $E[X] = \mu$ and var$(X) = \sigma^2$.

Geometric pmf: $P(X = k) = p(1-p)^{k-1}$, where $k = 1, 2, \ldots$

Exponential pdf: $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

Binomial pmf: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \ldots, n.$

Characteristic function: $\Phi_X(\omega) = E[e^{j\omega X}], \quad \omega \in \mathbb{R}.$

Conditional expectation: $E[h(Y)] = E_X\{E_Y[h(Y)|X]\}.$

Jacobian: If $v = g_1(x, y)$ and $w = g_2(x, y)$, then

$$J(v,w) = \det \begin{bmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{bmatrix} \text{ and } J(x,y) = \det \begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \end{bmatrix}$$

1. A game of tic-tac-toe (or noughts and crosses) is being played between two computers and with three squares left, the board is as shown in Figure 1. The two computers

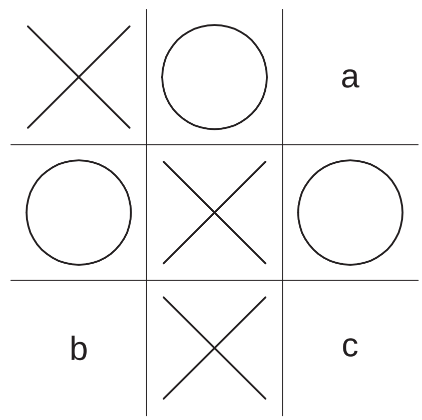

Figure 1: Tic-tac-toe board with three moves left.

(programmed by MIT students) can only pick moves purely at random, with a uniform distribution over all remaining squares. For instance, if the computer playing $\times$ were to move next, there is an equal probability of 1/3 of its placing a $\times$ in squares a, b or c. As a student at U of T, well known for its outstanding foundational courses in probability, you have been asked by MIT to evaluate their program by solving the following problems.

   (a) If the next move belongs to $\times$, find the probability of $\times$ winning the game. (6 marks)

   (b) If the next move belongs to $\bigcirc$, find the probabilities of the following events: "$\times$ wins the game", "$\bigcirc$ wins the game", "game is tied". (14 marks)

2. Let $Y = 2X$ where X is a random variable with an unspecified distribution.

   (a) Show that the correlation coefficient between X and Y, $\rho_{X,Y}$, has a value of 1 regardless of the distribution of X. (4 marks)

   (b) Now let X be exponentially distributed with parameter $\lambda = 1$. Find the joint cdf of X and Y, $F(x, y)$. (Hint: Let $Y = 2X$ in the definition of the joint cdf.) (10 marks)

   (c) Using the joint cdf in part (b), compute the probability of (X, Y) lying in the rectangle defined by the points (0.5,1), (0.5,3), (1.5,1) and (1.5,3). Noting that Y can be obtained from X, give an alternative method of finding this probability using only the pdf of X. (6 marks)

3. Let R and Θ be independent random variables with the following marginal probability density functions:

$$f_R(r) = \begin{cases} 1 & 0 < r \le 1 \\ 0 & \text{otherwise,} \end{cases} \qquad f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

Now we define $X = R\cos\Theta$ and $Y = R\sin\Theta$.

(a) Find the joint pdf of R and Θ. Be sure to indicate the regions of validity of your expression(s). (4 marks)

(b) Show that the Jacobian of the transformation from (R, Θ) to (X, Y) is $J(r, \theta) = r$. (5 marks)

(c) Find the joint pdf of X and Y. (6 marks)

(d) Using only the marginal distributions of R and Θ, find $E[X^2 + Y^2]$ and $E[XY]$. (5 marks)

4. Let $W = X + Y$, where X and Y are independent discrete random variables with the following probability mass functions:

k	$P(X = k)$	j	$P(Y = j)$
-1	0.125	0	0.25
0	0.5	1	0.25
1	0.375	2	0.5

Find the pmf of W using the following two methods:

(a) List all possible values of W, find the equivalent events in (X, Y) for $\{W = w\}$, and hence find $P(W = w)$. (8 marks)

(b) Find the characteristic functions of X and Y ($\Phi_X(\omega)$ and $\Phi_Y(\omega)$), then obtain the characteristic function of W, $\Phi_W(\omega)$, and use $\Phi_W(\omega)$ to find the pmf of W. (12 marks)

Question 5 is on the next page...

5. You are competing in a pie tossing contest, where you toss a pie as high as possible. To get an edge, you first jump up to a distance X meters and then toss the pie while in the air for a further distance Y meters. So, your pie flies a total of $(X+Y)$ meters up.

   (a) If X and Y are independent random variables, with X having a mean of 1 and Y having a mean of 10, what is the expected value of your pie's maximum flying height $E[X+Y]$? (4 marks)

   (b) If X and Y are dependent (e.g. your arm gets weaker the higher you jump), with X having a mean of 1 and a variance of 0.1, but such that given X, Y has a mean of $(10-2X-2X^2)$, what is $E[X+Y]$ now? (10 marks)

   (c) Being the smart engineer that you are, you develop a pie with N booster rockets – after you toss it as high as you can, the pie uses a first stage booster rocket to climb even higher. It then uses a second boosting stage, and a third boosting stage, and so on up to N stages.

   Assume that the height added in stage k is X_k, and that $X_1, X_2, \ldots, X_N$ are i.i.d. random variables, each with a mean of μ and variance σ^2. Using the Central Limit Theorem, and assuming that N is large, find the (approximate) probability that the total extra height added by the N stages exceeds $N\mu + 2\sigma\sqrt{N}$, *in terms of the Q function*, i.e. do not attempt to *evaluate* the Q function. (6 marks)

End of Paper