UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, DECEMBER 2001

Third Year Computer Engineering Program

ECE302F – PROBABILITY AND APPLICATIONS

Exam Type: C

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Instructions:

- Answer all SIX (6) questions;
- There are FIVE (5) pages in this question booklet;
- All questions carry equal marks.

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1. Please provide BRIEF answers to the following questions. (Marks will be deducted for long answers!)

   (a) Event A occurs with probability $P[A]$ and event B occurs with probability $P[B]$. If A and B are equivalent events, express $P[A]$ in terms of $P[B]$. (2 marks)

   (b) If 37% of the accidents are caused by drunk drivers and 63% are caused by sober (not drunk) drivers, we conclude that the probability of causing accidents decreases with drinking. Explain why this is incorrect. (2 marks)

   (c) A fourth-year computer engineering student applies for graduate admission to three top schools: Cartoon, Youobscene and Snakehead. Her chances of getting in are 80%, 70% and 90% respectively, and the 3 schools consider her application independently. What is the probability that she gets into at least one of the three schools? (2 marks)

   (d) If the *continuous* random variable X is measured in units of kilograms, what units would we ascribe to the pdf $f_X(x)$? (Hint: Consider the normalization condition.) (2 marks)

(e) A sum process is defined by $S_n = X_1 + X_2 + ... + X_n$ where X_i is an iid sequence. Explain why the sum process is not stationary. (2 marks)

(f) Consider the joint bivariate distribution $f_{X,Y}(x, y)$. The expression $f_{X,Y}(x, y)\, dxdy$ refers to the probability over what region? (2 marks)

(g) Assume that the chances of having a boy or a girl are equal (50/50). A family has two children, one of which is a girl. What is the probability that the other child is also a girl? (2 marks)

(h) When is the variance of a sum of random variables equal to the sum of the individual variances? (2 marks)

(i) In a shuffled deck of 52 cards there are 4 aces. Two cards are drawn from the deck without replacement. What is the probability that both cards are aces? (2 marks)

(j) A random experiment is repeated a large number of times and the occurrence of events A and B is noted. How would you test whether events A and B are independent? (2 marks)

2. The input to a communication system is X and its output is Y. Both are discrete random variables with the sample space $S = \{0, 1, 2, 3\}$ and X is uniformly distributed, i.e. $P(X = x) = 0.25$, for $x = 0, 1, 2, 3$. Ideally, $Y = X$ but noise in the system can cause decision errors (i.e. $Y \neq X$).

A table of *conditional probabilities* $P(Y = y|X = x)$ is provided below.

		y			
		0	1	2	3
	0	0.9	0.1	0	0
x	1	0	0.9	0.05	0.05
	2	0	0	0.8	0.2
	3	0.1	0	0.1	0.8

(a) Given that $P(Y = 2|X = 0) = 0$, is it possible to obtain the output $Y = 2$ when $X = 0$? Noting that some of the entries in the table are zero, count the total number of ways for an error to be made. (3 marks)

(b) Find the probability of error i.e. $P(Y \neq X)$. (7 marks)

(c) Compute the marginal probability mass function of Y, i.e. $P(Y = y)$, $y = 0, 1, 2, 3$. (7 marks)

(d) Explain why each row of the table sums to 1, while the columns do not. (3 marks)

3. Let each of the n random variables $X_1, \ldots, X_n$ be Bernoulli distributed, with probability of success p, and assume that they are independent.

   (a) Find the *exact* probability mass function of the discrete random variable $S_n = \sum_{i=1}^{n} X_i$. (5 marks)

   (b) Find the mean and variance of S_n *without* using the pmf obtained in part (a). (5 marks)

   (c) By the central limit theorem, the approximation

   $$P[S_n = k] \approx P\left[k - \frac{1}{2} < Y \leq k + \frac{1}{2}\right]$$

   holds when n is large, where Y is a Gaussian random variable with mean and variance equal to those of S_n. Express this Gaussian approximation of $P[S_n = k]$ in terms of n, p, k and the Q function $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left[-\frac{x^2}{2}\right] dx$. (5 marks)

   (d) Using the Gaussian approximation, find the probability that S_n lies *outside* the range $\{0, \ldots, n\}$ in terms of n, p and the Q function. What does this probability value converge to as n grows to infinity? What does this result tell you about the accuracy of the Gaussian approximation as a function of n? (5 marks)

4. Let X_1 and X_2 be random variables with joint pdf

   $$f_{X_1,X_2}(x_1, x_2) = \begin{cases} \frac{1}{\beta^2} e^{-(x_1+x_2)/\beta}, & 0 \leq x_1,\ 0 \leq x_2,\ \beta = \text{constant} \\ 0, & \text{otherwise} \end{cases}$$

   and consider the following transformation

   $$\begin{aligned} Z_1 &= X_1 + X_2 \\ Z_2 &= \frac{X_1}{X_1 + X_2} \end{aligned} .$$

   (a) It is clear from the definition of $f_{X_1,X_2}(x_1, x_2)$ that the pdf is non-zero only for the region $0 \leq x_1 < \infty$ and $0 \leq x_2 < \infty$. For what region will $f_{Z_1,Z_2}(z_1, z_2)$ be non-zero? (Hint: If $X_2 = 0$ then $Z_2 = 1$. But note that X_2 is always greater than or equal to zero.) (6 marks)

(b) Derive $f_{Z_1,Z_2}(z_1, z_2)$. Recall that the Jacobians of the forward and inverse transformations are respectively

$$J(x_1, x_2) = \det \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} \end{bmatrix}, \quad J(z_1, z_2) = \det \begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{bmatrix}.$$

(10 marks)

(c) If your derivation in part (b) is correct, you should note from your solution that

$$f_{Z_1,Z_2}(z_1, z_2) = f_{Z_1}(z_1).$$

Thus, the joint pdf is equal to the marginal pdf over Z_1. What is the marginal pdf over Z_2? (2 marks)

(d) Are Z_1 and Z_2 independent random variables? (2 marks)

5. Dr. Evil is sending a secret number using 2 random variables X_1 and X_2. This number is equal to $E[X_1X_2]$. Your mission is to try and find this number as follows:

   (a) After many sleepless nights, you have found out that the average value of X_1 given $X_2 = x_2$ is (x_2^2+1). In other words, $E[X_1|X_2 = x_2] = (x_2^2+1)$. You then find that X_2 is simply a uniform random variable between 0 and 1. What is the secret number? (Hint: $E[X_1X_2|X_2 = x_2] = E[X_1x_2|X_2 = x_2] = x_2E[X_1|X_2 = x_2]$.) (7 marks)

   (b) Now, in addition to the information in part (a), you also find that, given $X_2 = x_2$, X_1 is a continuous uniform random variable in the range $[2 - a, 2]$, where a is a constant. Find a and the joint pdf of X_1 and X_2. (Hint: Note that a may be a function of x_2.) (10 marks)

   (c) Being as devious as he is, Dr. Evil realizes that you have found the secret number and changes the joint pdfs of X_1 and X_2. You realize that he is using two independent uniform random variables between 0 and 1. What is the new secret number? (3 marks)

6. The autocorrelation of a zero mean random process $X(t)$ is given by:

$$R_X(t_1, t_2) = e^{-|t_2-t_1|} \cdot \cos^2\left[2\pi(t_2^2 - t_1^2)\right]$$

   (a) Is $X(t)$ wide-sense stationary? (3 marks)

(b) At what times is $X(t)$ uncorrelated with $X(0)$? (3 marks)

(c) What is the average power (i.e. $E[X^2(t)]$) of $X(t)$? (3 marks)

(d) A zero mean noise process $N(t)$ is added to $X(t)$ with autocorrelation:

$$R_N(t_1, t_2) = c \cdot e^{-|t_2 - t_1|} \cdot \left\{1 + \sin^2\left[2\pi(t_2^2 - t_1^2)\right]\right\}$$

where c is a constant. If this noise has the same power as $X(t)$ and is independent at all times of $X(t)$, then what is the autocorrelation of $X(t) + N(t)$? (Hint: You need to find c first.) (8 marks)

(e) Is $X(t) + N(t)$ wide-sense stationary? (3 marks)