ECE221S QUIZ No.1B **Wednesday, January 29, 2003**

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(1) Consider a *spherical* coordinate system (r,θ,ϕ) in which point P is located at (50m, 60°, 30°) and point P' is located at (42m, 30°, 45°). The position vectors of P and P' in the *spherical* coordinate system are **r** = $50\hat{\mathbf{a}}_r$ [m] and **r'** = $42\hat{\mathbf{a}}_r$ [m], respectively. Find

- **(a)** the position vectors **r** and **r'** in the *Cartesian* coordinate system.
- **(b)** the distance vector **R** = **r** – **r'** in the *Cartesian* coordinate system.
- **(c)** the distance vector **R** = **r** – **r'** at point P in the *spherical* coordinate system
- **(d)** the distance vector **R** = **r** – **r'** at point P' in the *spherical* coordinate system

Hint: Make use of the Table given below.

(2) A circular ring of radius *a* [m] carries a uniform charge ρ_L [C/m] and is placed on the xy-plane with axis the same as the *z*-axis. **Without invoking symmetry**, i.e. only by making use of vector calculus, show that in a *cylindrical* coordinate system the ρ-component of the electric field at points on the axis of the ring (z-axis) is zero.

(1) Solution:[1]

(a) In the *Cartesian* coordinate system:

P = [50sin(60°)cos(30°), 50sin(60°)sin(30°), 50cos(60°)] = (37.50, 21.65, 25.00)

$$\mathbf{r} = 37.50\hat{\mathbf{a}}_x + 21.65\hat{\mathbf{a}}_y + 25.00\hat{\mathbf{a}}_z \text{ [m]}$$

P' = [42sin(30°)cos(45°), 42sin(30°)sin(45°), 42cos(30°)] = (14.85, 14.85, 36.37)

$$\mathbf{r}' = 14.85\hat{\mathbf{a}}_x + 14.85\hat{\mathbf{a}}_y + 36.37\hat{\mathbf{a}}_z \text{ [m]}$$

(b) In the *Cartesian* coordinate system:

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = 37.50\hat{\mathbf{a}}_x + 21.65\hat{\mathbf{a}}_y + 25.00\hat{\mathbf{a}}_z - 14.85\hat{\mathbf{a}}_x - 14.85\hat{\mathbf{a}}_y - 36.37\hat{\mathbf{a}}_z$$

$$= 22.65\hat{\mathbf{a}}_x + 6.80\hat{\mathbf{a}}_y - 11.37\hat{\mathbf{a}}_z \text{ [m]}$$

$$= R_x\hat{\mathbf{a}}_x + R_y\hat{\mathbf{a}}_y + R_z\hat{\mathbf{a}}_z \text{ [m]}$$

(c) In the *spherical* coordinate system:

At P:

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = [+R_x\sin(\theta)\cos(\phi) + R_y\sin(\theta)\sin(\phi) + R_z\cos(\theta)]\hat{\mathbf{a}}_r$$
$$[+R_x\cos(\theta)\cos(\phi) + R_y\cos(\theta)\sin(\phi) - R_z\sin(\theta)]\hat{\mathbf{a}}_\theta$$
$$[-R_x\sin(\phi) + R_y\cos(\phi)]\hat{\mathbf{a}}_\phi$$

[1] See textbook for relationship between vector components in Cartesian, cylindrical and spherical coordinate systems.

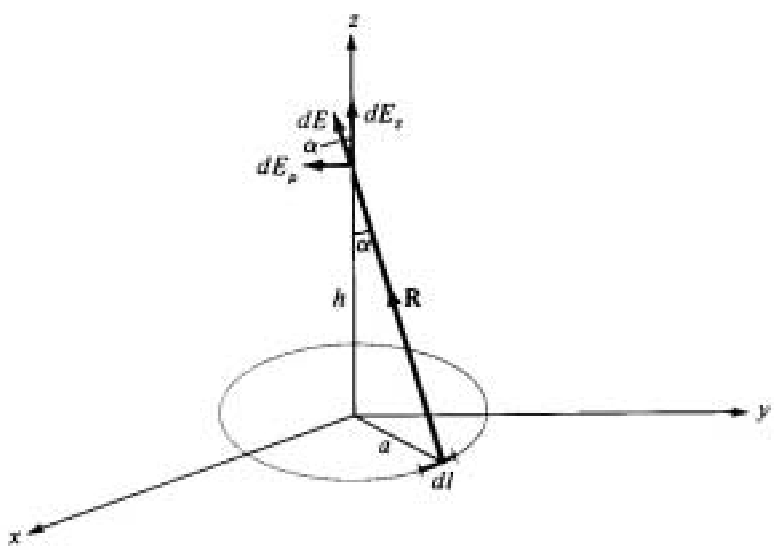
$$= [+22.65\sin(60°)\cos(30°) + 6.80\sin(60°)\sin(30°) - 11.37\cos(60°)]\hat{\mathbf{a}}_r$$
$$[+22.65\cos(60°)\cos(30°) + 6.80\cos(60°)\sin(30°) + 11.37\sin(60°)]\hat{\mathbf{a}}_\theta$$
$$[-22.65\sin(30°) + 6.80\cos(30°)]\hat{\mathbf{a}}_\phi$$
$$= [16.99 + 2.94 - 5.69]\hat{\mathbf{a}}_r + [9.81 + 1.70 + 9.85]\hat{\mathbf{a}}_\theta + [-11.33 + 5.89]\hat{\mathbf{a}}_\phi \text{ [m]}$$
$$= 14.24\hat{\mathbf{a}}_r + 21.36\hat{\mathbf{a}}_\theta - 5.44\hat{\mathbf{a}}_\phi \text{ [m]}$$

(d) In the *spherical* coordinate system:

At P':

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = [+ R_x \sin(\theta)\cos(\phi) + R_y \sin(\theta)\sin(\phi) + R_z\cos(\theta)]\,\hat{\mathbf{a}}_r$$
$$[+ R_x\cos(\theta)\cos(\phi) + R_y\cos(\theta)\sin(\phi) - R_z \sin(\theta)]\,\hat{\mathbf{a}}_\theta$$
$$[- R_x \sin(\phi) \qquad + R_y \cos(\phi)]\,\hat{\mathbf{a}}_\phi$$
$$= [+22.65\sin(30°)\cos(45°) + 6.80\sin(30°)\sin(45°) - 11.37\cos(30°)]\hat{\mathbf{a}}_r$$
$$[+22.65\cos(30°)\cos(45°) + 6.80\cos(30°)\sin(45°) + 11.37\sin(30°)]\hat{\mathbf{a}}_\theta$$
$$[-22.65\sin(45°) + 6.80\cos(45°)]\hat{\mathbf{a}}_\phi$$
$$= [8.01 + 2.40 - 9.85]\hat{\mathbf{a}}_r + [13.87 + 4.16 + 5.69]\hat{\mathbf{a}}_\theta + [-16.02 + 4.81]\hat{\mathbf{a}}_\phi \text{ [m]}$$
$$= 0.56\hat{\mathbf{a}}_r + 23.72\hat{\mathbf{a}}_\theta - 11.21\hat{\mathbf{a}}_\phi \text{ [m]}$$

(2) Solution:



(a) $\mathbf{E}(0,0,h) = \int \frac{\rho_L dl}{4\pi\varepsilon_o R^2}\,\hat{\mathbf{a}}_\mathbf{R}$ where: $\hat{\mathbf{a}}_\mathbf{R} = \mathbf{R} / R$

$\mathbf{R} = -a\,\hat{\mathbf{a}}_\rho + h\,\hat{\mathbf{a}}_z$

$= \frac{\rho_L}{4\pi\varepsilon_o}\int \frac{dl}{R^3}\mathbf{R}$

$R = |\mathbf{R}| = (a^2 + h^2)^{1/2}$

$dl = a\,d\phi$

$$= \frac{\rho_L}{4\pi\varepsilon_o} \int_0^{2\pi} \frac{a\, d\phi}{\left(a^2 + h^2\right)^{3/2}} \, (-a\, \hat{\mathbf{a}}_\rho + h\, \hat{\mathbf{a}}_\mathbf{z})$$

$$= \frac{-\rho_L}{4\pi\varepsilon_o} \int_0^{2\pi} \frac{a^2}{\left(a^2 + h^2\right)^{3/2}} \, \hat{\mathbf{a}}_\rho \, d\phi + \frac{\rho_L}{4\pi\varepsilon_o} \int_0^{2\pi} \frac{a\ h}{\left(a^2 + h^2\right)^{3/2}} \, \hat{\mathbf{a}}_\mathbf{z} \, d\phi$$

$$= \frac{-\rho_L a^2}{4\pi\varepsilon_o \left(a^2 + h^2\right)^{3/2}} \int_0^{2\pi} \hat{\mathbf{a}}_\rho \, d\phi + \frac{\rho_L a h}{4\pi\varepsilon_o \left(a^2 + h^2\right)^{3/2}} \, \hat{\mathbf{a}}_\mathbf{z} \int_0^{2\pi} d\phi$$

Note: The unit vector $\hat{\mathbf{a}}_\rho$ changes its direction with ϕ so it must be properly integrated. This is the reason why it is not taken outside the integration sign. However, the unit vector $\hat{\mathbf{a}}_\mathbf{z}$ can be taken out because it does not change its direction with angle ϕ.

Since $\hat{\mathbf{a}}_\rho = \cos\phi\, \hat{\mathbf{a}}_\mathbf{x} + \sin\phi\, \hat{\mathbf{a}}_\mathbf{y}$, the above may be rewritten as:

$$\mathbf{E}(0,0,h) = \frac{-\rho_L a^2}{4\pi\varepsilon_o \left(a^2 + h^2\right)^{3/2}} \left[\int_0^{2\pi} \cos\phi\, \hat{\mathbf{a}}_\mathbf{x} \, d\phi + \int_0^{2\pi} \sin\phi\, \hat{\mathbf{a}}_\mathbf{y} \, d\phi \right] + \frac{\rho_L a h}{4\pi\varepsilon_o \left(a^2 + h^2\right)^{3/2}} \, \hat{\mathbf{a}}_\mathbf{z} \, \phi\Big|_0^{2\pi}$$

$\hat{\mathbf{a}}_\mathbf{x}$ and $\hat{\mathbf{a}}_\mathbf{y}$ do not change directions as ϕ varies, so they can be moved outside the integration:

$$\mathbf{E}(0,0,h) = \frac{-\rho_L a^2}{4\pi\varepsilon_o \left(a^2 + h^2\right)^{3/2}} \left[\hat{\mathbf{a}}_\mathbf{x} \int_0^{2\pi} \cos\phi\, d\phi + \hat{\mathbf{a}}_\mathbf{y} \int_0^{2\pi} \sin\phi\, d\phi \right] + \frac{\rho_L a h}{4\pi\varepsilon_o \left(a^2 + h^2\right)^{3/2}} \, \hat{\mathbf{a}}_\mathbf{z} \, 2\pi$$

$$= \frac{-\rho_L a^2}{4\pi\varepsilon_o \left(a^2 + h^2\right)^{3/2}} \left[\hat{\mathbf{a}}_\mathbf{x} \sin\phi\Big|_0^{2\pi} - \hat{\mathbf{a}}_\mathbf{y} \cos\phi\Big|_0^{2\pi} \right] + \frac{\rho_L a h}{2\varepsilon_o \left(a^2 + h^2\right)^{3/2}} \, \hat{\mathbf{a}}_\mathbf{z}$$

$$= \frac{-\rho_L a^2}{4\pi\varepsilon_o \left(a^2 + h^2\right)^{3/2}} \left[\hat{\mathbf{a}}_\mathbf{x} 0 - \hat{\mathbf{a}}_\mathbf{y} 0 \right] + \frac{\rho_L a h}{2\varepsilon_o \left(a^2 + h^2\right)^{3/2}} \, \hat{\mathbf{a}}_\mathbf{z}$$

$$= \frac{\rho_L a h}{2\varepsilon_o \left(a^2 + h^2\right)^{3/2}} \, \hat{\mathbf{a}}_\mathbf{z} = \frac{\rho_L a h}{2\varepsilon_o \left(h^2 + a^2\right)^{3/2}} \, \hat{\mathbf{a}}_\mathbf{z} \quad \text{[V/m]} \qquad (\textit{as expected!})$$

Obviously, contributions along $\hat{\mathbf{a}}_\rho$ add up to zero. However, the result was obtained without invoking symmetry.