ECE221S Midterm **Tuesday, March 11th 03**

Family Name **Given Name**	**Student Number**

Please Insert Your **Name in Full and Student Number**

$$\nabla\cdot\mathbf{D}=\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z} \qquad \nabla\cdot\mathbf{D}=r_v \qquad \oint_S \mathbf{D}.d\mathbf{S} = Q_{encl}, \qquad \int r\sin(ar)dr = \frac{1}{a^2}\sin(ar)-\frac{r}{a}\cos(ar)$$

(1) (a) A spherical surface of radius 3mm is centered at P(4m,1m,5m) in free space. Let $\mathbf{D} = x\mathbf{a}_x$ C/m^2. Find the net electric flux leaving the spherical surface. **(2 marks)**

Since **D** is electric flux density, the flux leaving the surface is $\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla\cdot\mathbf{D}dv$. The last step uses divergence theorem. Using the equations given above, one could also use Gauss' law

$\oint_S \mathbf{D}.d\mathbf{S} = Q_{encl} = \int_V r_v dv = \int_V \nabla\cdot\mathbf{D}dv$. Also from the equations given, $\nabla\cdot\mathbf{D}=\frac{\partial D_x}{\partial x}=1$

$\therefore \oint_S \mathbf{D}.d\mathbf{S} = 1\times(\text{volume of sphere}) = \frac{4}{3}p\left(3\times10^{-3}\right)^3 = 36p\times10^{-9}\,C$. Note the units are Coulombs (C).

(b) Find the electric field everywhere given an infinite cylinder of charge with **(3 marks)**

$\rho_v = 0$	$\rho < 1$m,
$\rho_v = 2\sin(2000p\,\rho)$	1m $< \rho < 1.5$m,
$\rho_v = 0$	$\rho > 1.5$m

Clearly, this problem is solved using Gauss' law. Due to symmetry, the fields are in the r-direction. As done several times in class, the Gaussian surface will be a cylinder of radius r and length L. We need to worry about three cases, $r < 1$, $1 < r < 1.5$ and $r > 1.5$.

Gauss' Law : $\oint_S \mathbf{D}.d\mathbf{S} = Q_{encl}$. Also, $\mathbf{D} = D_r\,\mathbf{a}_r \Rightarrow \oint_S \mathbf{D}.d\mathbf{S} = D_r\,2pr\,L$

Case 1: $r < 1$. Since the charge density for $r < 1$ is zero, there is no charge enclosed within any surface. Therefore the flux density **D** = 0 and **E** = 0.

Case 3: $r > 1.5$. Now, the surface encloses all the charge within the length L.

$$Q_{encl}=\int_V r_v dv=\int_0^L\int_0^{2p}\int_1^{1.5}2\sin(2000pr')\,dr'\,df'\,dz'=2pL\int_1^{1.5}2\sin(2000pr')dr'$$

$$=4pL\left[\frac{1}{(2000p)^2}\sin(2000pr')-\frac{r'}{2000p}\cos(2000pr')\right]\Bigg|_{r'=1}^{r'=1.5}$$

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$$Q_{encl} = 4pL\left\{\frac{1}{(2000p)^2}\left[\sin(3000p) - \sin(2000p)\right] - \frac{1.5}{2000p}\cos(3000p) + \frac{1}{2000p}\cos(2000p)\right\}$$

Since sin(*2np*) = 0 and cos(*2np*) = 1 for integer *n*, sin(3000*p*) = sin(2000*p*)=0, cos(3000*p*)=cos(2000*p*)=1.

$$Q_{encl} = \frac{-0.5}{2000p}4pL \Rightarrow D_r = \frac{-1}{(2000p)r}, \qquad \mathbf{D} = \left[\frac{-1}{(2000p)r}\right]\mathbf{a}_r .$$

Case 2: 1 < *r* < 1.5. Now, the Gaussian surface encloses only some of the charge.

$$Q_{encl} = \int_V r_v dv = \int_0^L \int_0^{2p} \int_1^r 2\sin(2000pr')\, dr'\, df'\, dz' = 2pL\int_1^r 2\sin(2000pr')dr'$$

$$= 4pL\left[\frac{1}{(2000p)^2}\sin(2000pr') - \frac{r'}{2000p}\cos(2000pr')\right]\Bigg|_{r'=1}^{r'=r}$$

$$= 4pL\left\{\frac{1}{(2000p)^2}\left[\sin(2000pr) - \sin(2000p)\right] - \frac{r}{2000p}\cos(2000pr) + \frac{1}{2000p}\cos(2000p)\right\}$$

$$= 4pL\left[\frac{\sin(2000pr)}{(2000p)^2} - \frac{r}{2000p}\cos(2000pr) + \frac{1}{2000p}\right]$$

$$D_r = \frac{2}{r}\left[\frac{\sin(2000pr)}{(2000p)^2} - \frac{r}{2000p}\cos(2000pr) + \frac{1}{2000p}\right]$$ and **D** = $D_r\, \mathbf{a}_r$.

SOLUTION

(2) (a)

$$V_A = \frac{Q}{4\pi\epsilon_0 |\vec{r}_A - \vec{r}_B'|} \quad \text{WITH } V=0 \text{ AT } \infty \text{ AS REF.}$$

$$= \frac{200\pi\epsilon_0}{4\pi\epsilon_0 \left|(5\vec{a}_x + 6\vec{a}_y + 7\vec{a}_z) - (3\vec{a}_x - 1\vec{a}_y + 2\vec{a}_z)\right|}$$

$$= \frac{50}{|2\vec{a}_x + 7\vec{a}_y + 5\vec{a}_z|} = \frac{50}{\sqrt{2^2 + 7^2 + 5^2}}$$

$$= \frac{50}{\sqrt{4 + 49 + 25}} = \frac{50}{\sqrt{78}} = \frac{50}{8.831760866}$$

$$= 5.661385171 \text{ V} \approx 5.66 \text{ V AT } A(5m, 6m, 7m)$$

$$V_C = \frac{Q}{4\pi\epsilon_0 |\vec{r}_C - \vec{r}_B'|} \quad \text{WITH } V=0 \text{ AT } \infty \text{ AS REF.}$$

$$= \frac{200\pi\epsilon_0}{4\pi\epsilon_0 \left|(0\vec{a}_x + 0\vec{a}_y + 1\vec{a}_z) - (3\vec{a}_x - 1\vec{a}_y + 2\vec{a}_z)\right|}$$

$$= \frac{50}{|-3\vec{a}_x + 1\vec{a}_y - 1\vec{a}_z|} = \frac{50}{\sqrt{(-3)^2 + 1^2 + (-1)^2}}$$

$$= \frac{50}{\sqrt{9 + 1 + 1}} = \frac{50}{\sqrt{11}} = \frac{50}{3.31662479}$$

$$= 15.07556723 \text{ V} \approx 15.08 \text{ V AT } C(0m, 0m, 1m)$$

$$V_{AC} = V_A - V_C = 5.66\text{ V} - 15.08\text{ V}$$

$$= \underline{\underline{-9.42 \text{ V}}} \quad \text{WITH } V=0 \text{ AT } C \text{ AS REF.}$$

SOLUTION

(2)(b) $V_A = -\frac{\rho_L}{2\pi\epsilon_0} \ln \rho$ WITH $V=0$ AT $\rho=1$ AS REF.

$= -\frac{40\pi\epsilon_0}{2\pi\epsilon_0} \ln\left(\sqrt{6^2+7^2}\right)$

$= -20 \ln\left(\sqrt{36+49}\right) = -20 \ln\left(\sqrt{85}\right)$

$= -20 \ln(9.219544457)$

$= -20 \times 2.221325628$

$= -44.42651256 \approx -44.43\,V$ AT A

$V_C = -\frac{\rho_L}{2\pi\epsilon_0} \ln \rho$ WITH $V=0$ AT $\rho=1$ AS REF.

$= -\frac{40\pi\epsilon_0}{2\pi\epsilon_0} \ln\left(\sqrt{0^2+1^2}\right)$

$= -20 \ln(1)$

$= 0\,V$ AT C

$V_{AC} = V_A - V_C = -44.43V - 0V$

$= \underline{\underline{-44.43V}}$ WITH $V=0$ AT C AS REF.

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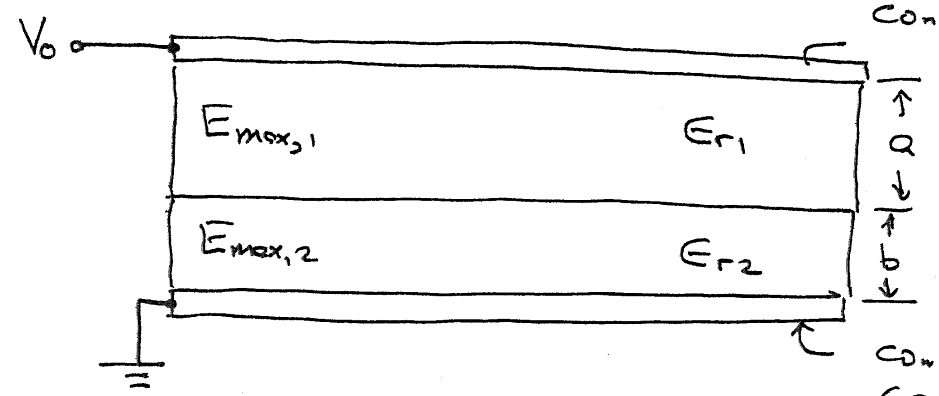
QUESTION 3

a)

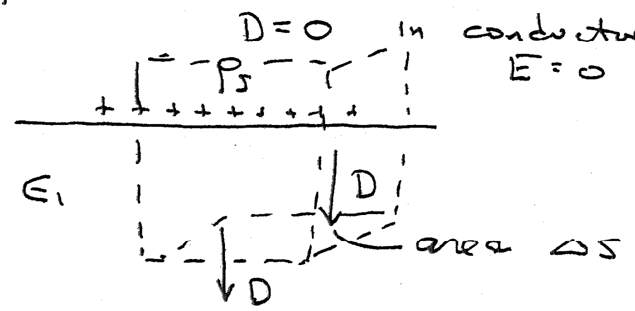


$a, b << c, d$

$\Rightarrow$ 1-D geometry

$\Rightarrow$ 2 layer parallel plate capacitor

b) To determine the electric field in the dielectric regions, it is convenient to use Gauss's Law. The potential V_0 establishes a uniform surface charge density $+\rho_s$ on the lower surface of the upper plate



$$\oint \bar{D} \cdot d\bar{S} = Q_{encl}$$

$$D \Delta S = \rho_s \Delta S$$

$$D = \rho_s$$

Gauss's Law shows that

$$\bar{D}_1 = \rho_s (-\hat{a}_z)$$

$$\bar{D}_2 = \rho_s (-\hat{a}_z)$$

But $\bar{E} = \bar{D}/\epsilon$ $\therefore$ $\overrightarrow{E_1} = \dfrac{\rho_s}{\epsilon_0 \epsilon_{r1}} (-\hat{a}_z)$ Mat'l #1

$$\overline{E_2} = \frac{\rho_s}{\epsilon_0 \epsilon_{r2}} (-\hat{a}_z) \quad \text{Mat'l \#2}$$

The question requires $\bar{E}_1, \bar{E}_2$ in terms of V_0, a, b etc
Using

$$V_0 - 0 = -\int \bar{E} \cdot d\bar{l} = -\int_0^b \bar{E}_2 \cdot d\bar{l} - \int_b^{b+a} \bar{E}_1 \cdot d\bar{l}$$

$$V_0 = \frac{\rho_s b}{\epsilon_0 \epsilon_{r2}} + \frac{\rho_s a}{\epsilon_0 \epsilon_{r1}}$$

$$\rho_s = \frac{V_0 \epsilon_0}{\frac{a}{\epsilon_{r1}} + \frac{b}{\epsilon_{r2}}} = \frac{V_0 \epsilon_0 \epsilon_{r1} \epsilon_{r2}}{a \epsilon_{r2} + b \epsilon_{r1}}$$

and
$$\boxed{\bar{E}_1 = \frac{V_0 \epsilon_{r2}}{a \epsilon_{r2} + b \epsilon_{r1}} (-\hat{a}_z)} \qquad \boxed{\bar{E}_2 = \frac{V_0 \epsilon_{r1}}{a \epsilon_{r2} + b \epsilon_{r1}}}$$

c) $a = 3\text{mm} \rightarrow 3 \times 10^{-3}$ m
$\epsilon_{r1} = 10$
$E_{max,1} = 70 \times 10^6$ V/m

$b = 1\text{mm} \rightarrow 10^{-3}$ m
$\epsilon_{r2} = 6$
$E_{max,2} = 30 \times 10^3$ V/m

$$a\epsilon_{r2} + b\epsilon_{r1} = (3 \times 6 + 1 \times 10) 10^{-3} = 28 \times 10^{-3}$$

V_0 – potential difference across structure
$V_{0,m1}$ – pot. diff such that $E_1 = E_{max,1}$

$$V_{0,m1} = \frac{70 \times 10^6 \times 28 \times 10^{-3}}{6} = 326.7 \text{ kV}$$

$$V_{0,m2} = \frac{30 \times 10^6 \times 28 \times 10^{-3}}{10} = 84.0 \text{ kV}$$

Clearly, if V_0 is increased from 0, the max. potential difference that can be applied before either region breaks down is

$$\boxed{V_{0,max} = 84\ kV}$$

(4)

(a) $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$ $\rho_v = -\frac{\rho_0}{r^2}$ $\boxed{\nabla^2 V = \frac{\rho_0}{\varepsilon_0 r^2}}$

(b) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \frac{\rho_0}{\varepsilon_0 r^2}$ $\underset{r \neq 0}{\rightsquigarrow}$ $\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \frac{\rho_0}{\varepsilon_0}$ $\underset{\text{integrate}}{\rightsquigarrow}$ $r^2 \frac{\partial V}{\partial r} = \frac{\rho_0}{\varepsilon_0} r + C_1$

$\underset{r \neq 0}{\rightsquigarrow}$ $\frac{\partial V}{\partial r} = \frac{\rho_0}{\varepsilon_0}\frac{1}{r} + \frac{C_1}{r}$ $\underset{\text{integrate}}{\rightsquigarrow}$ $\boxed{V(r) = \frac{\rho_0}{\varepsilon_0}\ln(r) - \frac{C_1}{r^2} + C_2}$

Boundary conditions: $V(a) = V_0$ $V(b) = 0$

$$\left.\begin{aligned} \frac{\rho_0}{\varepsilon_0}\ln(a) - \frac{C_1}{a} + C_2 &= V_0 \\ \frac{\rho_0}{\varepsilon_0}\ln(b) - \frac{C_1}{b} + C_2 &= 0 \end{aligned}\right\} \quad \frac{\rho_0}{\varepsilon_0}\ln(a/b) + C_1\left(\frac{1}{b} - \frac{1}{a}\right) = V_0$$

$$\boxed{C_1 = -\frac{V_0 + \frac{\rho_0}{\varepsilon_0}\ln(b/a)}{1/a - 1/b}}$$

$$C_2 = \frac{C_1}{b} - \frac{\rho_0}{\varepsilon_0}\ln(b) = -\frac{\left(V_0 + \frac{\rho_0}{\varepsilon_0}\ln(b/a)\right)/b}{1/a - 1/b} - \frac{\rho_0}{\varepsilon_0}\ln(b)$$

$$\boxed{C_2 = -\frac{V_0/b + \frac{\rho_0}{\varepsilon_0}\left(\ln(b)/a - \ln(a)/b\right)}{1/a - 1/b}}$$

$$\boxed{V(r) = \frac{\rho_0}{\varepsilon_0}\ln(r) + \frac{V_0 + \frac{\rho_0}{\varepsilon_0}\ln(b/a)}{1/a - 1/b}\,\frac{1}{r} - \frac{V_0/b) + \frac{\rho_0}{\varepsilon_0}\left(\ln(b)/a - \ln(a)/b\right)}{1/a - 1/b}}$$

(c) $\underline{E} = -\nabla V = -\hat{\underline{r}}\,\frac{\partial V}{\partial r}$

$$\boxed{\underline{E}(r) = \left\{-\frac{\rho_0}{\varepsilon_0}\frac{1}{r} + \frac{V_0 + \frac{\rho_0}{\varepsilon_0}\ln(b/a)}{1/a - 1/b}\,\frac{1}{r^2}\right\}\hat{\underline{r}}}$$