Question #1

Let $\alpha \in \mathbf{R}$ and $\beta \in \mathbf{R}$ be positive. Let $S(\alpha,\beta)$ be a family of sets in $\mathbf{R}^2$ defined as follows:

$$S(\alpha,\beta) \;=\; \{(x,y) \mid -\alpha \le x \le \alpha, -\beta \le y \le \beta\}$$

1. Sketch and label the set $S(1,1)$ in $\mathbf{R}^2$.
2. What is $\bigcup_{\alpha>0,\beta>0} S(\alpha,\beta)$?
3. What is $\bigcap_{\alpha>0,\beta>0} S(\alpha,\beta)$?

[**10 Marks**]

Solution:

1. $S(\alpha,\beta)$ is a rectangle centred at $(0,0)$ with width 2α and height 2β.
   Thus, $S(1,1)$ is a square of width 2 centered at $(0,0)$. [**3 Marks**]
2. The union of all rectangles centred at $(0,0)$ is $\mathbf{R}^2$. [**3 Marks**]
3. The intersection of all rectangles centred at $(0,0)$ is $\{(0,0)\}$. [**4 Marks**]

Question #3

Show that $((p \wedge q) \rightarrow r)$ and $(p \rightarrow (q \rightarrow r))$ are logically equivalent.

[**10 Marks**]

Solution:

p	q	r	$p \wedge q$	$((p \wedge q) \rightarrow r)$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	T	T	T
F	T	F	F	T	F	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

[**10 Marks**]

Question #8

The following functions are defined on $\mathbf{R} \to \mathbf{R}$.

- State whether they are *one-to-one* and whether they are *onto*.
- Find the best Big-O estimate for each of them.

1. $f(x) = x^3 + \sin(x) - 3$

2. $g(x) = \dfrac{1}{1+e^{-x}}$

[15 Marks]

Solution:

1. $f(x) = 2x^3 + \sin(x) - 3$
   It is *one-to-one*. **[2 Marks]**
   It is *onto*. **[2 Marks]**
   $f(x) = O(x^3)$. **[3 Marks]**

2. $g(x) = \dfrac{1}{1+e^{-x}}$
   It is *one-to-one*. **[2 Marks]**
   It is not *onto*. **[2 Marks]**
   $g(x) = O(1)$. **[4 Marks]**

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Question #2

Using the *rules of inference*, what *logical conclusion* (of no more than 12 words), can be drawn from all of the following hypotheses:

i) If John is well, he has not eaten a green apple.
ii) If John is not well, he calls a doctor.
iii) If John's life is safe, he will not call a doctor.

Give the details of your argument.

[10 Marks]

Solution:

Let $p :=$ John eats a green apple.
$q :=$ John calls a doctor.
$s :=$ John is safe

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Given: $q \to \neg p$, $\neg q \to r$, $s \to \neg r$ 1

$(q \to \neg p) \iff (p \to \neg q)$ (contrapositive)

$(p \to \neg q) \wedge (\neg q \to r) \Rightarrow (p \to r)$ (Hypothetical Syllogism)

$(s \to \neg r) \iff (r \to \neg s)$ (contrapositive) 4

$\therefore$ $(p \to r) \wedge (r \to \neg s) \Rightarrow (p \to \neg s)$ (Hypothetical Syllogism)

Conclusion:

If John eats a green apple then his life is not safe. 2

Note: Marks will not be deducted for not giving the names of the rules of inference.

(10)

Question #4

Consider the following sentence:

"Every natural number is the sum of the squares of two natural numbers."

(i) State this sentence as a *quantified predicate*.

(ii) What is the negation of this *predicate*? (Please move the negation inside all quantifiers.)

(iii) State *in words* the negation of the sentence.

[10 Marks]

Solution :

(i) Let the universe of discourse for all variables be "N", the set of natural numbers, then the statement is

$$\forall n \; \exists x \; \exists y \; [n = x^2 + y^2] \qquad 4$$

where $n, x, y \in N$

(ii) $\neg \forall n \; \exists x \; \exists y \; [n = x^2 + y^2]$

$$\Longleftrightarrow \exists n \; \forall x \; \forall y \; [\neg (n = x^2 + y^2)] \qquad 3$$

$$\Longleftrightarrow \exists n \; \forall x \; \forall y \; [n \neq x^2 + y^2]$$

(iii) There is a natural number that cannot be written as the sum of the squares of two natural numbers.

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Question #5

State and **prove** the *"First Form of the Theorem on Mathematical Induction"*.
Hint: You may use the *Well-Ordering Property*. **[15 Marks]**

Solution:

Let $S(1), S(2), \ldots$ be a sequence of propositions having the following two properties:

(i) $S(1) := T$, (Basis Property)

(ii) $S(n) := T$ materially implies $S(n+1) := T$ for $n \geq 1$, (Inductive Property)

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Then $S(n) := T$ for every positive integer n.

Proof: Let $p := S(1) := T$

$q :=$ if $S(n) := T$, then $S(n+1) := T$

$r := \forall n \; S(n) := T$ (where $D_n := \mathbb{Z}^+$)

Assume $\neg r := T$, hence $\exists n \; S(n) := F$ $(n \in \mathbb{Z}^+)$

i.e. there is $G := \{ n \,;\, S(n) := F \} \neq \phi$

Let $\check{n}$ denote the smallest positive integer in the set G, then $\check{n} = 1$ or $\check{n} > 1$.

case 1: If $\check{n} = 1$, then $S(1) := F$

$\therefore \neg r \rightarrow \neg p$

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case 2: If $\check{n} > 1$, then $S(\check{n} - 1) := T$

$\therefore \neg r \rightarrow \neg q$

Hence, $\neg r \rightarrow (\neg p \vee \neg q)$

$\therefore \; p \wedge q \rightarrow r$ (contrapositive)

Q. E. D.

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Question #6

Prove that for all positive integers n,

$$\sum_{j=n}^{2n-1}(2j+1) = 3n^2. \qquad \textbf{[15 Marks]}$$

Solution.

Basis Step: For $n=1$

$$\sum_{j=1}^{1}(2j+1) = 3 = 3 \times 1^2 = 3 \quad \text{is True}$$

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Inductive Step: Assume that $S(n) := \sum_{j=n}^{2n-1}(2j+1) = 3n^2$

$$S(n+1) := \sum_{j=n+1}^{2(n+1)-1}(2j+1)$$

$$= \left[\sum_{j=n}^{2n-1}(2j+1)\right] - (2n+1) + (4n+1) + (4n+3)$$

$$= 3n^2 - (2n+1) + (4n+1) + (4n+3)$$

$$= 3n^2 + 6n + 3$$

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$$= 3(n+1)^2$$

$\therefore$ $S(n) \to S(n+1)$ is True

$\therefore$ $\sum_{j=n}^{2n-1}(2j+1) = 3n^2$ for all positive integers n.

Q.E.D.

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Question #7

Consider the binary relation R on the set of real numbers.

$R := \{(x,y) \mid x - y \textit{ is an even integer}\}$.

i) Show that R is an equivalence relation on the set of real numbers.

ii) What are the equivalence classes of 1 and $\frac{1}{2}$ with respect to R. **[15 Marks]**

Solution

i) Since $x - x = 0$ is an even integer for every $x \in R$

$\therefore$ R is reflexive. 2

If $(x,y) \in R$ then $x-y$ is an even integer,
then $-(x-y) = y-x$ is also an even integer.
$\therefore$ R is symmetric. 2

If $(x,y) \in R$ and $(y,z) \in R$ then $(x-y)$ & $(y-z)$ are even integers. Since $x - z = (x-y) + (y-z)$ and the sum of two even integers is an even integer
$\therefore$ R is transitive. 4

Hence, R is an equivalence relation. 1

ii) $E(1) := \{ y \mid (1-y) \text{ is an even integer}\}$
$= \{ y \mid y = 1 - 2k \text{ where } k \text{ is an integer}\}$ 3
$= \{ \dots, -3, -1, 1, 3, 5, \dots \}$

$E(\frac{1}{2}) := \{ y \mid (\frac{1}{2} - y) \text{ is an even integer}\}$
$= \{ y \mid y = \frac{1}{2} - 2k \text{ where } k \text{ is an integer}\}$
$= \{ \dots, -\frac{7}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \dots \}$ 3